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A new approach to the Darboux–Bäcklund transformation versus the standard dressing method

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Abstract

We present a new approach to the construction of the Darboux matrix. This is a generalization of a recently formulated method based on the assumption that the square of the Darboux matrix vanishes for some values of the spectral parameter. We consider the multisoliton case, the reduction problem and the discrete case. The relationships between our approach, the Zakharov–Shabat dressing method and the Neugebauer–Meinel method are discussed in detail.

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1. Introduction

There are several methods to construct the Darboux matrix (which generates soliton solutions) [1–8]. However, these methods are technically difficult when applied to the matrix versions of the spectral problems which are naturally represented in Clifford algebras [9, 10, 12]. Some of these problems are avoided in our recent paper [13]. In the present paper we develop the ideas of [13] in the matrix case. We extend our approach to the multisoliton case and consider the reduction problem and the discrete case. We also show that our approach, although different, is to some extent equivalent to the standard dressing method. We compare our method with the Zakharov–Shabat approach [1, 14] and the Neugebauer–Meinel approach [3, 15].

We consider the spectral problem

$$\Psi_{,\mu} = U_\mu \Psi \qquad (\mu = 1, \ldots, m) \tag{1}$$

where U_μ depend on $x^1, \ldots, x^m, \lambda$. We make no assumptions on U_μ except a given rational dependence on λ (i.e. the poles of U_μ are prescribed). The Darboux transformation is defined as the gauge-like transformation

$$\tilde{\Psi} = D\Psi, \tag{2}$$

leading to a new solution of the spectral problem (1)

$$\tilde{\Psi}_{,\mu} = \tilde{U}_\mu \tilde{\Psi}, \tag{3}$$

which means that $\tilde{U}_\mu$ and U_μ should have the same rational dependence on λ. The compatibility conditions for system (1) yield a system of nonlinear equations for the coefficients of U_μ. The Darboux transformation automatically generates new solutions to this nonlinear system. In the simplest case (D linear in λ or D with a single pole in λ) the Darboux transformation usually adds a soliton on a given background.

In this paper, we assume that U_μ and Ψ are $n \times n$ matrices but our approach also works well in the Clifford numbers case [13].

The construction of the Darboux transformation is well known (especially in the matrix case) [7, 14]. The first step is the equation for D resulting from (1), (2) and (3):

$$D_{,\mu} + DU_\mu = \tilde{U}_\mu D. \tag{4}$$

In our earlier paper [13] we proposed the following procedure. We assume that there exist two different values of λ, say λ_+ and λ_-, satisfying

$$D^2(\lambda_\pm) = 0. \tag{5}$$

Denoting $\Psi(\lambda_\pm) = \Psi_\pm$, $D(\lambda_\pm) = D_\pm$, evaluating (4) at $\lambda = \lambda_\pm$ and multiplying (4) by $D_\pm$ from the right, we get

$$D_{\pm,\mu} D_\pm + D_\pm U_\mu(\lambda_\pm) D_\pm = 0. \tag{6}$$

We assume that $\Psi(\lambda_\pm)$ are invertible (which is true in the generic case). It is not difficult to check that $D_\pm$ given by

$$D_\pm = \varphi_\pm \Psi_\pm d_\pm \Psi_\pm^{-1}, \qquad d_\pm^2 = 0, \tag{7}$$

(where $d_\pm =$ const and $\varphi_\pm$ are scalar functions) satisfy equations (5), (6). Assuming that D is linear in λ, i.e.

$$D(\lambda) = A_0 + A_1 \lambda, \tag{8}$$

we can easily express A_0, A_1 by $D_\pm$ to get

$$D(\lambda) = \frac{\lambda - \lambda_-}{\lambda_+ - \lambda_-} \varphi_+ \Psi_+ d_+ \Psi_+^{-1} + \frac{\lambda - \lambda_+}{\lambda_- - \lambda_+} \varphi_- \Psi_- d_- \Psi_-^{-1}. \tag{9}$$

2. One-soliton case and the Zakharov–Shabat approach

We confine ourselves to the case linear in λ (see (8)). Condition (5) can be easily realized if

$$D^2(\lambda) = \sigma(\lambda - \lambda_+)(\lambda - \lambda_-) I, \tag{10}$$

where $\sigma \neq 0$ is a constant, $\lambda_+ \neq \lambda_-$ and I is the identity matrix. The identity matrix will sometimes be omitted (i.e. for $a \in \mathbf{C}$ we write $aI = a$). In case (10) from (5) and (9) it follows that

$$D_+ D_- + D_- D_+ = -\sigma(\lambda_+ - \lambda_-)^2. \tag{11}$$

Lemma 1. *D of the form (8) satisfies (10) if and only if n is even and*

$$D = \mathcal{N}\left(\lambda - \lambda_+ + (\lambda_+ - \lambda_-) P\right), \tag{12}$$

where the matrices $\mathcal{N}$ and P satisfy

$$P^2 = P, \qquad \mathcal{N}^2 = \sigma, \qquad \mathcal{N}P\mathcal{N}^{-1} = I - P. \tag{13}$$

In this case the Darboux matrices (9) and (12) are equivalent.

Proof. We assume (8) and identify $\mathcal{N} \equiv A_1$. Then

$$D^2(\lambda) = A_0^2 + (A_0\mathcal{N} + \mathcal{N}A_0)\lambda + \mathcal{N}^2\lambda^2,$$

i.e. $D^2(\lambda)$ is a quadratic polynomial. It is proportional to the identity matrix I (compare (10)) iff

$$\mathcal{N}^2 = \sigma, \qquad A_0\mathcal{N} + \mathcal{N}A_0 = -\sigma(\lambda_+ + \lambda_-), \qquad A_0^2 = \sigma\lambda_+\lambda_-. \tag{14}$$

Multiplying the second equation by $\mathcal{N}A_0$ we get

$$\sigma^2\lambda_+\lambda_- + (\mathcal{N}A_0)^2 + \sigma(\lambda_+ + \lambda_-)\mathcal{N}A_0 = 0.$$

Hence $(\mathcal{N}A_0 + \sigma\lambda_+)(\mathcal{N}A_0 + \sigma\lambda_-) = 0$, and, denoting $Q := \mathcal{N}A_0 + \sigma\lambda_+$, we have

$$Q^2 = (\lambda_+ - \lambda_-)\sigma Q$$

which means that $Q = (\lambda_+ - \lambda_-)\sigma P$, where $P^2 = P$. Therefore, taking into account $\mathcal{N}^2 = \sigma$, we get (12). Now, we take into account the third equation of (14). First, $A_0^2 P = \sigma\lambda_+\lambda_- P$ yields $\lambda_-(\lambda_+ - \lambda_-)\mathcal{N}P\mathcal{N}P = 0$. Then the equation $A_0^2 = \sigma\lambda_+\lambda_-$ is equivalent to $\lambda_+(\lambda_+ - \lambda_-)(\sigma(I - P) - \mathcal{N}P\mathcal{N}) = 0$. Therefore $\mathcal{N}P\mathcal{N}^{-1} = I - P$. This equality means that $\ker P = \mathcal{N}^{-1}\,\mathrm{im}\,P$, which implies $\dim\ker P = \dim\mathrm{im}\,P$. Thus, n is even which completes the proof. □

The case $\lambda_+ = \lambda_-$ can be treated in a similar way and it leads to the nilpotent case [7]:

$$D = \mathcal{N}(\lambda - \lambda_+ + M), \qquad M^2 = 0, \quad \mathcal{N}^2 = \sigma, \quad M = -\mathcal{N}M\mathcal{N}^{-1}.$$

Our method is closely related to the standard dressing transformation [1, 7, 14]. The Darboux matrix (12) can be rewritten as

$$D = (\lambda - \lambda_+)\mathcal{N}\left(I + \frac{\lambda_+ - \lambda_-}{\lambda - \lambda_+}P\right). \tag{15}$$

We recognize the standard one-soliton Darboux matrix in the Zakharov–Shabat form [7, 14]. We point out that usually one considers the Darboux matrix $\mathcal{D} = (\lambda - \lambda_+)^{-1}D$, which is equivalent to D given by (12) because the multiplication of D by a constant factor leaves equation (4) invariant [16]. $\mathcal{N}$ is known as the normalization matrix and P is a projector expressed by the background wavefunction:

$$\ker P = \Psi(\lambda_+)V_{\ker}, \qquad \mathrm{im}\,P = \Psi(\lambda_-)V_{\mathrm{im}}, \tag{16}$$

where $V_{\ker}$ and V_{im} are some constant vector spaces, λ_+ and λ_- are constant complex parameters. The last constraint of (13) has the following interpretation. Let $\mathcal{N}P\mathcal{N}^{-1} = I - P$. Then

$$v \in \mathrm{im}\,P \quad\Leftrightarrow\quad (I - P)v = 0 \quad\Leftrightarrow\quad P\mathcal{N}^{-1}v = 0 \quad\Leftrightarrow\quad \mathcal{N}^{-1}v \in \ker P$$

$$v \in \ker P \quad\Leftrightarrow\quad Pv = 0 \quad\Leftrightarrow\quad P\mathcal{N}^{-1}v = \mathcal{N}^{-1}v \quad\Leftrightarrow\quad \mathcal{N}^{-1}v \in \mathrm{im}\,P.$$

Hence, $\dim\mathrm{im}\,P = \dim\ker P = d \equiv n/2$, which implies $\dim V_{\mathrm{im}} = \dim V_{\ker}$. In this case, given a projector P, one can always find a corresponding $\mathcal{N}$. Indeed, let $v_1, \ldots, v_d$ be a basis in $\mathrm{im}\,P$ and $w_k := \mathcal{N}^{-1}v_k$ $(k = 1, \ldots, d)$ an associated basis in $\ker P$. By virue of $\mathcal{N}^2 = \sigma$ we have $\mathcal{N}^{-1}w_k = \sigma^{-1}v_k$. Therefore

$$\mathcal{N}^{-1}(v_1, \ldots, v_d, w_1, \ldots, w_d) = (w_1, \ldots, w_d, v_1/\sigma, \ldots, v_d/\sigma)$$

(where $(v_1, v_2, \ldots)$ denotes the matrix with columns $v_1, v_2, \ldots$) and, finally,

$$\mathcal{N} = (v_1, \ldots, v_d, w_1, \ldots, w_d)(w_1, \ldots, w_d, v_1/\sigma, \ldots, v_d/\sigma)^{-1}. \tag{17}$$

The $\mathcal{N}$ obtained in this way depends on the choice of the bases $v_1, \ldots, v_d$ and $w_1, \ldots, w_d$ (we can put Av_k, $\det A \neq 0$, in the place of v_k and Bw_j, $\det B \neq 0$, in the place of w_j). In other words, $\mathcal{N}$ is given up to non-degenerate $d \times d$ matrices A and B.

Formulae (9) and (12) coincide after appropriate identification of the parameters. Indeed, comparing coefficients by powers of λ we have

$$\begin{aligned} \mathcal{N} &= \frac{\varphi_+\Psi_+ d_+\Psi_+^{-1} - \varphi_-\Psi_- d_-\Psi_-^{-1}}{\lambda_+ - \lambda_-}, \\ \mathcal{N}(-\lambda_+ + (\lambda_+ - \lambda_-)P) &= \frac{\lambda_+\varphi_-\Psi_- d_-\Psi_-^{-1} - \lambda_-\varphi_+\Psi_+ d_+\Psi_+^{-1}}{\lambda_+ - \lambda_-}, \end{aligned} \tag{18}$$

and after straightforward computation we get

$$\begin{aligned} P &= \left(\varphi_+\Psi_+ d_+\Psi_+^{-1} - \varphi_-\Psi_- d_-\Psi_-^{-1}\right)^{-1} \varphi_+\Psi_+ d_+\Psi_+^{-1}, \\ I - P &= \left(\varphi_-\Psi_- d_-\Psi_-^{-1} - \varphi_+\Psi_+ d_+\Psi_+^{-1}\right)^{-1} \varphi_-\Psi_- d_-\Psi_-^{-1}. \end{aligned} \tag{19}$$

Taking into account assumption (11) we have

$$P = \frac{D_- D_+}{D_+ D_- + D_- D_+} = \frac{-D_- D_+}{\sigma(\lambda_+ - \lambda_-)^2}. \tag{20}$$

The above results are valid for $n \times n$ matrix linear problems. Now, we focus on the 2×2 case. Because the elements d_+, d_- are nilpotent $\left(d_\pm^2 = 0\right)$, then there exist vectors v_+, v_- such that

$$d_+ v_+ = 0, \qquad d_- v_- = 0. \tag{21}$$

Then from (19) it follows immediately that $P\Psi_+ v_+ = 0$ and $(I - P)\Psi_- v_- = 0$, i.e. $\Psi_+ v_+$ span $\ker P$ and $\Psi_- v_-$ span $\operatorname{im} P$. Hence, $v_+ \in V_{\mathrm{ker}}$ and $v_- \in V_{\mathrm{im}}$.

It is not difficult to check that the general form of 2×2 matrices $d_\pm$ such that $d_\pm^2 = 0$ is given by

$$d_\pm = \begin{pmatrix} -a_\pm b_\pm & b_\pm^2 \\ -a_\pm^2 & a_\pm b_\pm \end{pmatrix} = \begin{pmatrix} b_\pm \\ a_\pm \end{pmatrix} \begin{pmatrix} -a_\pm & b_\pm \end{pmatrix}, \tag{22}$$

where $a_\pm, b_\pm$ are complex numbers. Therefore, to satisfy (21), we can take

$$v_+ = \begin{pmatrix} b_+ \\ a_+ \end{pmatrix}, \qquad v_- = \begin{pmatrix} b_- \\ a_- \end{pmatrix}. \tag{23}$$

We have almost unique correspondence (i.e. up to a scalar factor) between v_+ and d_+ and between v_- and d_-.

Denoting

$$\Psi_+ v_+ \equiv \begin{pmatrix} B_+ \\ A_+ \end{pmatrix}, \qquad \Psi_- v_- \equiv \begin{pmatrix} B_- \\ A_- \end{pmatrix},$$

we get the explicit formula for P

$$P = \begin{pmatrix} 0 & B_- \\ 0 & A_- \end{pmatrix} \begin{pmatrix} B_+ & B_- \\ A_+ & A_- \end{pmatrix}^{-1} = \frac{\begin{pmatrix} -A_+B_- & B_+B_- \\ -A_+A_- & B_+A_- \end{pmatrix}}{A_-B_+ - A_+B_-}. \tag{24}$$

The corresponding $\mathcal{N}$ reads (compare (17))

$$\mathcal{N} = \frac{1}{A_-B_+ - A_+B_-} \begin{pmatrix} \sigma A_-B_- - A_+B_+ & B_+^2 - \sigma B_-^2 \\ \sigma A_-^2 - A_+^2 & A_+B_+ - \sigma A_-B_- \end{pmatrix}. \tag{25}$$

Although we can reduce our approach to the explicit formulae (24) and (25) the main advantage of our method consists in expressing the Darboux transformation in terms of $\Psi_\pm d_\pm \Psi_\pm^{-1}$ and avoiding difficulties with parameterizing kernel and image of the projector P which is especially troublesome in the Clifford algebras case.

3. Reductions

Let us consider the unitary reduction

$$U_\mu^\dagger(\bar{\lambda}) = -U_\mu(\lambda). \tag{26}$$

If U_μ is a polynom in λ, then condition (26) means that the coefficients of this polynom by powers of λ are $u(n)$-valued.

One can easily prove that (26) implies $\Psi^\dagger(\bar{\lambda})\Psi(\lambda) = C(\lambda)$, where $C(\lambda)$ is a constant matrix ($C,_\nu = 0$). The matrix C can be fixed by a choice of the initial conditions. Usually we confine ourselves to the case

$$\Psi^\dagger(\bar{\lambda})\Psi(\lambda) = k(\lambda) I, \tag{27}$$

where $k(\lambda)$ is analytic in λ. From (27) we can derive $\overline{k(\bar{\lambda})} = k(\lambda)$. By virtue of (2), the Darboux matrix has to satisfy the analogical constraint:

$$D^\dagger(\bar{\lambda}) D(\lambda) = p(\lambda) I. \tag{28}$$

Assuming that D is a polynom with respect to λ, compare (8), we get that $p(\lambda)$ is a polynom with constant real coefficients, i.e. $\overline{p(\bar{\lambda})} = p(\lambda)$ and $p,_\nu = 0$.

Lemma 2. *If D is linear in λ and (28) holds, then roots of the equation* $\det D(\lambda) = 0$ *satisfy the quadratic equation* $p(\lambda) = 0$.

Proof. Let $p(\lambda) = \alpha\lambda^2 + \beta\lambda + \gamma$. From (8), (28) it follows that

$$A_0^\dagger A_0 = \gamma, \qquad A_1^\dagger A_1 = \alpha, \qquad A_0^\dagger A_1 + A_1^\dagger A_0 = \beta, \tag{29}$$

which can be easily reduced to a single equation for $S := -A_0 A_1^{-1}$. Namely,

$$\alpha S^2 + \beta S + \gamma = 0. \tag{30}$$

Therefore, the eigenvalues of S have to satisfy the equation $p(\lambda) = 0$. Indeed, if $S\vec{v} = \mu\vec{v}$, then $(\alpha\mu^2 + \beta\mu + \gamma)\vec{v} = 0$. On the other hand, the equation $\det D(\lambda) = 0$ can be rewritten as

$$0 = \det(\lambda I - S) \det A_1, \tag{31}$$

which means that the roots of $\det D(\lambda) = 0$ coincide with eigenvalues of S. □

Lemma 3. *We assume (10). Then reduction (27) imposes the following constraints on the Darboux matrix (9):*

$$\lambda_- = \bar{\lambda}_+, \qquad d_-^\dagger d_+ = 0, \tag{32}$$

and (for $n = 2$) $\langle v_+ | v_- \rangle = 0$.

In particular, by virtue of (5), we can take $d_- = f d_+^\dagger$, where f is a scalar function.

Proof. Let us denote zeros of the polynom $p(\lambda)$ by λ_1, λ_2. Because $\overline{p(\bar{\lambda})} = p(\lambda)$ there are two possibilities: either $\lambda_2 = \bar{\lambda}_1$ or λ_1, λ_2 are real. From (10) we have

$$(\det D(\lambda))^2 = \sigma^n (\lambda - \lambda_+)^n (\lambda - \lambda_-)^n. \tag{33}$$

Therefore, in case (10), lemma 2 means that λ_+, λ_- coincide with λ_1, λ_2.

Suppose that $\lambda_+ \in \mathbf{R}$. Then from (28) we have $(D(\lambda_+))^\dagger D(\lambda_+) = 0$, which implies $D_+ \equiv D(\lambda_+) = 0$ (because for any vector $v \in \mathbf{C}^n$ the scalar product $\langle v \mid D_+^\dagger D_+ v\rangle = 0$, hence $\langle D_+ v \mid D_+ v\rangle = 0$, and, finally $D_+ v = 0$). Therefore λ_+ (and, similarly, λ_-) cannot be real. Thus $\lambda_- = \bar{\lambda}_+$. In this case (28) reads

$$(D(\lambda_-))^\dagger D(\lambda_+) = 0. \tag{34}$$

Using (7) and (27) (assuming $k(\lambda_\pm) \neq 0$) we get

$$(D(\lambda_-))^\dagger = \bar{\varphi}_-(\Psi_-^\dagger)^{-1} d_-^\dagger \Psi_-^\dagger = \bar{\varphi}_- \Psi_+ d_-^\dagger \Psi_+^{-1}$$

and (34) assumes the form $\varphi_+ \bar{\varphi}_- \Psi_+ d_-^\dagger d_+ \Psi_+^{-1} = 0$. Hence $d_-^\dagger d_+ = 0$.

Finally, in the case $n = 2$, we use (22). Then the condition $d_-^\dagger d_+ = 0$ is equivalent to $a_+\bar{a}_- + b_+\bar{b}_- = 0$, i.e. $\langle v_+ \mid v_-\rangle = 0$. □

Another very popular reduction is given by

$$U_\mu(-\lambda) = J U_\mu(\lambda) J^{-1}, \qquad J^2 = c_0 I. \tag{35}$$

Then one can prove that $\Psi(-\lambda) = J\Psi(\lambda)C(\lambda)$, and we choose such initial conditions that $C(\lambda) = J^{-1}$, i.e.

$$\Psi(-\lambda) = J\Psi(\lambda)J^{-1}, \qquad D(-\lambda) = JD(\lambda)J^{-1}. \tag{36}$$

Such choice of $C(\lambda)$ is motivated by a natural requirement that Ψ, $\tilde{\Psi}$, D are elements of the same loop group (by the way, formula (27) has the same motivation).

Lemma 4. *We assume (10). Then reduction (36) imposes the following constraints on the Darboux matrix (9):*

$$\lambda_- = -\lambda_+, \qquad \varphi_+ = \varphi_-, \qquad d_+ = J^{-1} d_- J, \tag{37}$$

and (for $n = 2$) $v_- = Jv_+$.

Proof. From (36) it follows that $\det D(\lambda) = \det D(-\lambda)$, which means that the set of roots of the equation $\det D(\lambda) = 0$ is invariant under the transformation $\lambda \to -\lambda$. Therefore $\lambda_- = -\lambda_+$. Then, using once more (36) we get $D_- = JD_+J^{-1}$ and $\Psi_- = J\Psi_+J^{-1}$. Hence $\varphi_+ d_+ = \varphi_- J^{-1} d_- J$. Thus $\varphi_+ = c_0\varphi_-$, where c_0 is a constant. Without loss of generality we can take $c_0 = 1$ (redefining $d_\pm$ if necessary). In the case $n = 2$ the kernels of $d_\pm$ are one dimensional. Therefore $0 = d_+ v_+ = J^{-1} d_- J v_+$ implies $v_- = c_1 J v_+$, where $c_1 = \text{const}$. We can take $v_+ = Jv_-$. □

Other types of reductions (compare [2, 7]) can be treated in a similar way.

4. The multi-soliton Darboux matrix

In this section, we generalize the approach of [13]. First, we relax assumption (5). Second, we consider the N-soliton case (the Darboux matrix is a polynom of order N):

$$D(\lambda) = A_0 + A_1\lambda + \cdots + A_N\lambda^N. \tag{38}$$

Condition (5) will be replaced by

$$D(\lambda_k)T_k = 0, \tag{39}$$

where $T_k \neq 0$ are some matrices (for instance we can consider them as values of a matrix function $T(\lambda)$, i.e. $T_k \equiv T(\lambda_k)$, compare the end of this section). Elementary algebraic considerations show that the existence of a matrix T_k satisfying (39) is equivalent to $\det D(\lambda_k) = 0$.

Evaluating (4) at $\lambda = \lambda_k$, denoting $D_k \equiv D(\lambda_k)$, $\Psi_k \equiv \Psi(\lambda_k)$ and $U_{k\mu} \equiv U_\mu(\lambda_k)$, multiplying the resulting equation by T_k from the right and assuming that $\tilde{U}_{k\mu}$ is finite, we get

$$D_{k,\mu} T_k + D_k U_{k\mu} T_k = 0. \tag{40}$$

If D_k, T_k satisfy this equation, then $\tilde{U}_\mu(\lambda)$ is holomorphic (does not have a pole) at $\lambda = \lambda_k$. Therefore, in order to preserve the structure of U_μ, the coefficients A_j of the Darboux matrix (38) have to satisfy all equations resulting from (39) and (40) for any λ_k satisfying the equation $\det D(\lambda) = 0$.

To solve equation (40) we define d_k and h_k by

$$D_k = \Psi_k d_k \Psi_k^{-1}, \qquad T_k = \Psi_k h_k \Psi_k^{-1}. \tag{41}$$

Then

$$D_{k,\mu} = \Psi_{k,\mu} d_k \Psi_k^{-1} + \Psi_k d_{k,\mu} \Psi_k^{-1} - \Psi_k d_k \Psi_k^{-1} \Psi_{k,\mu} \Psi_k^{-1}.$$

Therefore

$$D_{k,\mu} = U_{k\mu} D_k + \Psi_k d_{k,\mu} \Psi_k^{-1} - D_k U_{k\mu},$$

and, taking into account (39) and (41), we rewrite (40) as follows:

$$\Psi_k d_{k,\mu} h_k \Psi_k^{-1} = 0. \tag{42}$$

Finally, as a straightforward consequence of (39) and (42) we get the following constraints on d_k and h_k:

$$d_k h_k = 0, \qquad d_k h_{k,\mu} = 0. \tag{43}$$

In [13] we confined ourselves to the case $N = 1$ and $T_k = D_k$ (in other words, $T(\lambda) = D(\lambda)$), i.e. $d_k = \varphi_k d_{0k}$ (φ_k scalar functions, d_{0k} constant elements satisfying $d_{0k}^2 = 0$), $h_k = d_k$. Now we obtain the general solution of (43) in the case of 2×2 matrices.

Lemma 5. *Let d and h are 2×2 matrices depending on $x^1, \ldots, x^m$ such that $dh = 0$, $dh,_\mu = 0$ and $d \neq 0, h \neq 0$. Then there exist constants c^1, c^2 and scalar functions q^1, q^2, p^1, p^2 (depending on $x^1, \ldots, x^m$) such that*

$$\begin{aligned} d &= \begin{pmatrix} q^1 c^2 & -q^1 c^1 \\ q^2 c^2 & -q^2 c^1 \end{pmatrix} = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} (c^2 \quad -c^1) \equiv q c^\perp, \\ h &= \begin{pmatrix} c^1 p^1 & c^1 p^2 \\ c^2 p^1 & c^2 p^2 \end{pmatrix} = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} (p^1 \quad p^2) \equiv c p^T. \end{aligned} \tag{44}$$

Proof. The columns of h are orthogonal to the rows of d. If $\det(d) \neq 0$, then, obviously, $h = 0$ in contrary to our assumptions. Therefore, $\det(d) = 0$ which means that the rows of d are linearly dependent. Similarly, the columns of h are linearly dependent as well. We denote them by $p^1 c$ and $p^2 c$ (where c is a column vector). Thus $h = c p^T$, where $p^T := (p^1, p^2)$.

$dh = 0$ means that the columns of h are orthogonal to the rows of d. Therefore, these rows are of the form $q^1 c^\perp, q^2 c^\perp$, where $c^\perp$ is a vector orthogonal to c, and, finally $d = q c^\perp$. Thus we obtained (44).

Taking into account the condition $dh,_\mu = 0$ we get

$$0 = q c^\perp (c,_\mu p^T + c p^T,_\mu) = q c^\perp c,_\mu p^T \quad \Rightarrow \quad c^\perp c,_\mu = 0.$$

This means that $c^2 c^1,_\mu = c^1 c^2,_\mu$, or c^2/c^1 is a constant. In other words, $c^1 = f c^{10}$, $c^2 = f c^{20}$ (f is a function, and c^{10}, c^{20} are constants). To complete the proof we redefine $p \to f p$, $q \to f q$ and $c^{k0} \to c^k$. □

Therefore,

$$D(\lambda_k) = \Psi(\lambda_k) q_k c_k^{\perp} \Psi^{-1}(\lambda_k), \tag{45}$$

where c_k are given constant column unit vectors, $c_k^{\perp}$ is a row vector orthogonal to c_k and q_k are some vector-valued functions (column vectors). We keep the notation $q_k c_k^{\perp} \equiv d_k$, but now in general $d_k^2 \neq 0$.

We notice that in the case $N = 1$ the freedom concerning the choice of q_1, q_2 corresponds to the arbitrariness of the normalization matrix. In particular, condition (5), which can be rewritten as $q_k = \varphi_k c_k$ ($k = 1, 2$ and φ_1, φ_2 are scalar functions), imposes strong constraints on $\mathcal{N}$ (see (18)).

Constraint (39) implies $\det D(\lambda_k) = 0$. In the case of 2×2 matrices the equation $\det D(\lambda) = 0$ (where D is given by (38)) has $2N$ roots (at most): $\lambda_1, \ldots \lambda_{2N}$.

Taking any $N + 1$ pairwise different roots (say $\lambda_1, \ldots, \lambda_{N+1}$) and using Lagrange's interpolation formula for polynomials, we get the generalization of formula (9):

$$D(\lambda) = \sum_{k=1}^{N+1} \left(\prod_{\substack{j=1 \\ j\neq k}}^{N+1} \frac{(\lambda - \lambda_j)}{(\lambda_k - \lambda_j)} \right) \Psi(\lambda_k) q_k c_k^{\perp} \Psi^{-1}(\lambda_k). \tag{46}$$

We also have $N - 1$ matrix constraints which result from evaluating formula (46) at $\lambda_{N+2}, \ldots, \lambda_{2N}$:

$$\sum_{k=0}^{N+1} \frac{\Psi(\lambda_k) q_k c_k^{\perp} \Psi^{-1}(\lambda_k)}{(\lambda_k - \lambda_0) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_{N+1})} = 0, \tag{47}$$

where $\lambda_0 = \lambda_{N+2}, \ldots, \lambda_{2N}$.

We denote

$$Q_k := \Psi(\lambda_k) q_k, \qquad C_k^{\perp} := c_k^{\perp} \Psi^{-1}(\lambda_k). \tag{48}$$

The Darboux matrix is parameterized by $2N$ constants λ_k, $2N$ vector functions q_k and $2N$ constant vectors c_k subject to constraint (47).

The crucial point consists in solving system (47) in order to get parameterization of the Darboux matrix by a set of independent quantities. We plan to express $2N - 2$ functions from among $Q_1, \ldots, Q_{2N}$ by other data. For instance, we choose Q_1, Q_2 as independent functions (they correspond to the normalization matrix $\mathcal{N}$).

We rewrite system (47) as

$$\sigma_{\nu 0} Q_\nu C_\nu^{\perp} + \sum_{k=1}^{N+1} \sigma_{\nu k} Q_k C_k^{\perp} = 0 \qquad (\nu = N + 2, \ldots, 2N) \tag{49}$$

where

$$\sigma_{\nu k} = \frac{1}{(\lambda_k - \lambda_\nu)(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_{N+1})}$$

$$\sigma_{\nu 0} = \frac{1}{(\lambda_\nu - \lambda_1) \cdots (\lambda_\nu - \lambda_N)(\lambda_\nu - \lambda_{N+1})}.$$

System (49) is linear with respect to Q_k. We are going to express $2N - 2$ vector functions $Q_3, \ldots, Q_{2N}$ by Q_1, Q_2 and by the other parameters: C_k, λ_k. Then, using (48), we could get $q_3, \ldots, q_{2N}$, etc. However, it is better to write (46) in terms of Q_k:

$$D(\lambda) = \sum_{k=1}^{N+1} \left(\prod_{\substack{j=1 \\ j\neq k}}^{N+1} \frac{(\lambda - \lambda_j)}{(\lambda_k - \lambda_j)} \right) Q_k c_k^{\perp} \Psi^{-1}(\lambda_k). \tag{50}$$

Taking the scalar product of (49) by C_1 we get

$$Q_\nu = -\sum_{k=2}^{N+1} \frac{\sigma_{\nu k}\langle C_k^\perp \mid C_1\rangle}{\sigma_{\nu 0}\langle C_\nu^\perp \mid C_1\rangle} Q_k \qquad (\nu = N+2, \ldots, 2N) \tag{51}$$

and the scalar product of the νth equation of (49) by C_μ yields

$$\sum_{k=1}^{N+1} \sigma_{\nu k}\langle C_k^\perp \mid C_\nu\rangle Q_k = 0 \qquad (\nu = N+2, \ldots, 2N). \tag{52}$$

This is a system of $N-1$ linear equations with respect to $Q_1, \ldots, Q_{N+1}$. Therefore, we can (for instance) express $Q_3, \ldots, Q_{N+1}$ in terms of Q_1, Q_2. Then, using (51), we have $Q_{N+2}, \ldots, Q_{2N}$ expressed in the similar way.

Our method is closely related to the Neugebauer–Meinel approach [3]. Let D is given by (38). We denote by $F(D(\lambda))$ the adjugate (or adjoint) matrix of D which is, obviously, a polynom in λ. Thus,

$$D(\lambda)F(D(\lambda)) = w(\lambda)I, \tag{53}$$

where $w(\lambda) = \det(D(\lambda))$ is a scalar polynom and I is the identity matrix. Therefore, we can put $T(\lambda) = F(D(\lambda))$ in formula (39) and identify λ_k with zeros of $\det D(\lambda)$.

In the Neugebauer approach the matrix coefficients A_k of the Darboux matrix (38) are obtained by solving the following system:

$$D(\lambda_k)\Psi(\lambda_k)c_k = 0 \qquad (k = 1, \ldots, nN), \tag{54}$$

where λ_k and constant vectors c_k are treated as given parameters. Thus, one has n^2N scalar equations for $(N+1)n^2$ scalar variables. One of the matrices A_k, say A_N, is considered as undetermined normalization matrix.

We point out that $D(\lambda_k)$ given by formula (45) satisfy (54).

5. The discrete case

The discrete analogue of (1) is the following system of linear difference equation

$$T_\mu\Psi = U_\mu\Psi \qquad (\mu = 1, \ldots, m), \tag{55}$$

where T_ν denotes the shift in the νth variable, i.e. $(T_\nu\Psi)(x^1, \ldots, x^\nu, \ldots, x^m) := \Psi(x^1, \ldots, x^\nu + 1, \ldots, x^m)$. The Darboux transformation is defined in the standard way:

$$\tilde{\Psi} = D\Psi, \qquad T_\mu\tilde{\Psi} = \tilde{U}_\mu\tilde{\Psi}. \tag{56}$$

Therefore $(T_\mu D)(T_\mu\Psi) = \tilde{U}_\mu D\Psi$, and, finally

$$(T_\mu D)U_\mu = \tilde{U}_\mu D. \tag{57}$$

If $D^2(\lambda_1) = 0$, then multiplying (57) by $D(\lambda)$ from the right, and evaluating the obtained equation at $\lambda = \lambda_1$ we see that the right-hand side vanishes and we get

$$(T_\mu D_1)U_\mu(\lambda_1)D_1 = 0, \tag{58}$$

where we denote $D_1 := D(\lambda_1)$. In order to solve (58) we put

$$D_1 = \varphi_1\Psi_1 d_1\Psi_1^{-1},$$

where $\Psi_1 := \Psi(\lambda_1)$. Then (58) takes the form

$$\varphi_1 T_\mu(\varphi_1)(T_\mu\Psi_1)(T_\mu d_1)d_1\Psi_1^{-1} = 0.$$

Therefore, if

$$(T_\mu d_1)d_1 = 0, \tag{59}$$

then equation (58) is satisfied. Condition (59) can be rewritten (at least in the matrix case) as

$$\operatorname{im} d_1 \subset \ker (T_\mu d_1).$$

In other words, the sequence of linear operators

$$\cdots \to T_\mu^{-1} d_1 \to d_1 \to T_\mu d_1 \to T_\mu^2 d_1 \to \cdots$$

is an exact sequence [17].

Similarly as in the smooth case the simplest solution of (59) is $d_1 = \text{const}$ such that $d_1^2 = 0$. Then the Darboux matrix has the same form (9) as in the continuum case.

6. Summary

In this paper, we developed the approach outlined in our earlier paper [13]. We extended its results on the N-soliton case and presented some preliminary results on the discrete case. It turned out that the discrete case is, as usual, very similar to the continuous one.

In the case $N = 1$ we considered explicitly the most important reductions. The results of section 2 show that the form (15) of the Darboux matrix, very convenient as far as the unitary reduction is concerned (compare [2]), is a necessary consequence of very weak assumptions (8) and (26). We point out also lemma 2, which states that in the case of the unitary reduction the equation $\det D(\lambda) = 0$ has exactly two roots provided that D is linear in λ. This fact is not obvious for matrices of higher dimensions n.

The approach of Neugebauer and Meinel [3] applied to 2×2 spectral problems produces determinants of the order $2N$. Our formula (50) contains (arbitrary) parameters λ_k, $c_k^\perp$ and matrices $Q_1, \ldots, Q_{N+1}$. These matrices satisfy system (52) of linear algebraic equations. The solution of (52) can be expressed in terms of determinants of the order $N - 1$. A similar, but apparently different, simplification was obtained earlier for $su(2)$-AKNS spectral problems [18].

The main motivation to develop the presented method is the construction of N-fold Darboux transformation for Spin-valued spectral problems in terms of Clifford numbers rather than matrices. It is already done in the case $N = 1$ (see [12, 13, 19]). To extend these results to $N > 1$ one has to solve equations (43) in the case of Clifford numbers, i.e. to find an appropriate generalization of lemma 5, and then to solve the resulting analogue of constraint (47). Both problems remain open.

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